

STICK INDEX OF KNOTS AND LINKS IN THE CUBIC LATTICE

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ABSTRACT. The cubic lattice stick index of a knot type is the least number of sticks necessary to construct the knot type in the 3-dimensional cubic lattice. We present the cubic lattice stick index of various knots and links, including all $(p, p+1)$ -torus knots, and show how composing and taking satellites can be used to obtain the cubic lattice stick index for a relatively large infinite class of knots. Additionally, we present several bounds relating cubic lattice stick index to other known invariants.

1. INTRODUCTION

An important invariant in physical knot theory is the stick index, namely the least number of sticks (i.e. line segments) needed to construct a given knot in 3-space. In this paper we are interested in a related invariant, the cubic lattice stick index, which is the least number of sticks (without regard to length) necessary to construct a knot in the 3-dimensional cubic lattice $(\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R})$. Since we do not refer to other lattices, we will usually omit the word *cubic*. Throughout, we will let K represent a fixed conformation of a knot and $[K]$ the collection of all conformations of a given knot type. We begin with some definitions:

Definition 1. A *stick* of a lattice stick conformation K is a maximal line segment of K in \mathbb{R}^3 , i.e. a line segment contained in K that is not contained in any longer line segment of K . An *x-stick* of K is a stick parallel to the x -axis. We define *y-sticks* and *z-sticks* similarly.

Definition 2. The *lattice stick number* $s_{CL}(K)$ of a lattice stick conformation K is the number of constituent sticks of K . We minimize $s_{CL}(K)$ over all lattice stick conformations $K \in [K]$ and we call this number the *lattice stick index* $s_{CL}[K]$ of the knot type $[K]$. A lattice

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stick conformation realizing this minimum is called a *minimal lattice conformation*.

Previously, lattice stick index was known for three knots. In 1993, Diao proved that $s_{CL}[3_1] = 12$ ([3]). Then in 1999, Promislow and Rensburg proved that $s_{CL}[9_{47}] = 18$ ([10]), and in 2005 Huh and Oh proved that $s_{CL}[4_1] = 14$ ([8]). Huh and Oh also proved in 2010 that the trefoil and figure eight are the only knots with lattice stick index less than 15 ([7]).

In Section 2 we present a lower bound on lattice stick index based on bridge number. We use this to find the lattice stick indices of 8_{20} , 8_{21} , and 9_{46} , as well as all $(p, p+1)$ -torus knots. In Section 3, we show how *exterior L's* and *clean L's* can be used to compose minimal lattice stick conformations to obtain the minimal lattice stick conformation of the composite knot. In Section 4, we show how known minimal lattice stick conformations can be used to determine lattice stick indices for various satellite knots. In Section 5, we present both upper and lower bounds on s_{CL} in terms of crossing number. Finally, in Section 6, we consider the minimal lattice stick index of certain links of more than one component.

2. DETERMINING CUBIC LATTICE STICK INDEX

We use a lower bound based on bridge index and an upper bound based on construction to determine the lattice stick index of many knots.

Definition 3. The **bridge index** of a knot is given by

$$b[K] = \min_{K \in [K]} \min_{\mathbf{v} \in S^2} (\# \text{ of local maxima when } K \text{ is projected to } \mathbf{v})$$

Bridge index was first introduced in [11] and has proved to be an extremely useful invariant. The following lemma was also found by Promislow and Rensburg in [10].

Lemma 2.1. $s_{CL}[K] \geq 6b[K]$.

Proof. Let $K \in [K]$ be a minimal lattice stick conformation. Then K must have at least $b[K]$ local maxima in each of the x, y and z directions. First considering the z direction, let S be a portion of K corresponding to a local z -maximum and contained in a certain z -level. Let s_1 and s_2 be the first z -sticks we encounter as we continue along K in either direction, starting at S . Since S corresponds to a local maximum, s_1 and s_2 must both connect the z -level of S to lower z -levels. Then for each local z -maximum of K we have two distinct

corresponding z -sticks, and it follows that K must contain at least $2b[K]$ z -sticks. A similar argument for the x and y directions shows that K must contain at least $2b[K]$ sticks in each of the three primary directions. \square

Note that by the same argument, we must have two z -sticks corresponding to each local z -minimum, and similarly for the y and z directions. Then if K is a lattice stick conformation constructed with exactly $6b[K]$ sticks, each stick must connect a local minimum to a local maximum in its own direction and lie at a local extremum in the other two primary directions.

Theorem 2.2. *Knots 8_{20} , 8_{21} , and 9_{46} all have lattice stick index 18.*

Proof. Knots 8_{20} , 8_{21} , and 9_{46} are all 3-bridge knots. Lemma 2.1 and Figure 1 complete the proof. \square

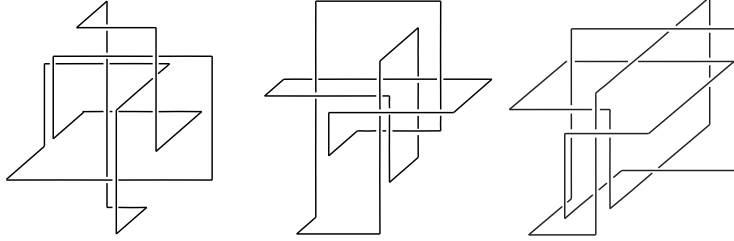


FIGURE 1. Minimal lattice conformations of 8_{20} , 8_{21} , and 9_{46} .

Next we find the lattice stick index for an infinite class of prime knots, namely the $(p, p+1)$ -torus knots, which we denote $T_{p,p+1}$.

Theorem 2.3. $s_{CL}[T_{p,p+1}] = 6p$ for $p \geq 2$.

Proof. By [11], we know $b[T_{p,p+1}] = p$. Figure 2 shows a construction of the first three knots in the class of $(p, p+1)$ -torus knots using $6p$ sticks. Using the pattern in Figure 2, we can construct the rest of the $(p, p+1)$ -torus knots with $6p$ sticks. Lemma 2.1 completes the proof. \square

3. COMPOSITION

We begin by defining clean L 's and exterior L 's of a lattice stick conformation.

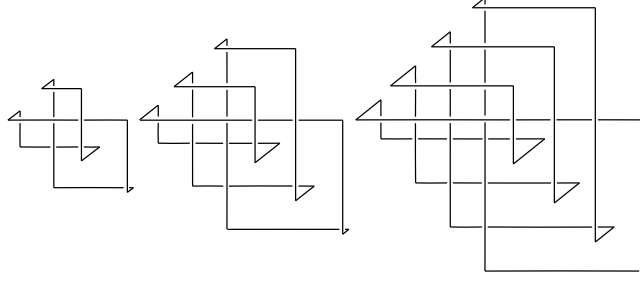


FIGURE 2. Minimal lattice conformations of $T_{2,3}$, $T_{3,4}$, and $T_{4,5}$.

Definition 4. Let S be a portion of a lattice stick conformation K consisting of four consecutive sticks s_1, s_2, s_3, s_4 , where s_1 and s_4 are parallel to each other and s_2 and s_3 are perpendicular both to each other and to s_1 and s_4 . Let R be the closed rectangle with two edges of which are given by s_2 and s_3 . If s_1 and s_4 lie to the same side of the plane containing R and $R \cap K \subset S$, we call S a *clean L* of K . If in addition K is entirely contained in one of the two closed half-spaces defined by the plane containing R , we call S an *exterior L*.

In [11], Schubert showed that $b[K_1 \# K_2] = b[K_1] + b[K_2] - 1$. We use this fact and Lemma 2.1 to prove the following theorem for lattice stick index under composition.

Theorem 3.1. *Let K_1 and K_2 be minimal lattice conformations realizing $s_{CL}[K_i] = 6b[K_i]$ for $i = 1, 2$ such that K_1 has an exterior L and K_2 has a clean L . Then $s_{CL}[K_1 \# K_2] = s_{CL}[K_1] + s_{CL}[K_2] - 6$.*

Proof. Let L_1 be an exterior L of K_1 and let L_2 be a clean L of K_2 . Let R_1 and R_2 be the closed rectangles defined by the middle two sticks of L_1 and L_2 respectively. Assume that R_2 lies at the level $z = 0$ with the first and last sticks of L_2 lying in the region $z \leq 0$. We now $(+z)$ -translate all vertices of K_2 lying in $z > 0$ by some large integer n_0 , and modify the sticks of K_2 accordingly so that K_2 is still a minimal lattice stick conformation of the same knot type, but with no x -sticks or y -sticks in the region $0 < z \leq n_0$. We call this move *expanding K_2 in the half-space $z \geq 0$ by n_0* .

Now let P_1, P_2, P_3, P_4 denote the four planes that are perpendicular to the plane containing R_2 and that contain an edge of R_2 . Let S_i denote the closed half-space defined by P_i that does not contain R_2 . Similar to before, for $n = 1, 2, 3, 4$ we expand the K_2 in S_i by n_i , where n_i is a sufficiently large integer.

The resulting K_2 after these five expansions is still a minimal lattice stick conformation of the same knot type. The advantage is that we are now in a position to place K_1 (i.e. rotate and translate it) such that it lies in $z \geq 0$ with R_1 lying at $z = 0$, and such that K_1 and K_2 are disjoint except possibly at L_1 and L_2 . By scaling up K_2 in either the x or y direction, we can make R_2 a square. We can then scale up K_1 (without it intersecting K_2 , since n_0, \dots, n_4 were chosen sufficiently large) such that R_1 is a square of the same size. After a possible rotation, $R_1 \cap K_1$ and $R_2 \cap K_2$ are now identical. We remove R_1 and R_2 and consolidate the remaining four sticks of L_1 and L_2 into two z -sticks (see Figure 3). The result is a lattice stick conformation of $[K_1 \# K_2]$ with $s_{CL}[K_1] + s_{CL}[K_2] - 6$ sticks, which is minimal by Lemma 2.1.

□

The following corollary follows by induction.

Corollary 3.2. *For K_1, \dots, K_n minimal lattice knot conformations realizing $s_{CL}[K_i] = 6b[K_i]$ for $i = 1, \dots, n$ each possessing one exterior L and at least one other clean L , $s_{CL}[K_1 \# \dots \# K_n] = s_{CL}[K_1] + \dots + s_{CL}[K_n] - 6(n - 1)$.*

4. LATTICE STICK INDEX OF CERTAIN SATELLITE KNOTS

In this section we find the cubic lattice stick index of certain n -string satellite knots. We show that if we are given a companion knot $[J]$ with $s_{CL}[J] = 6b[J]$, certain satellites of $[J]$ have cubic lattice stick index $6nb[J]$.

A *braid pattern* B is defined to be a conformation of any knot in braid form that can be contained in a standardly embedded solid torus such that any meridional disk of the torus intersects B at least n times, where n is the number of strands. To form a satellite knot, the solid torus containing the pattern is homeomorphically mapped to a neighborhood

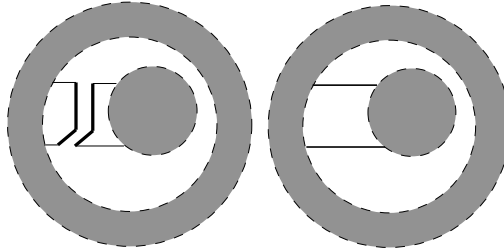


FIGURE 3. Gluing L_1 and L_2 by eliminating 6 sticks.

of the companion knot, carrying the pattern along. We say that a *braid satellite knot* is a satellite knot that has a braid pattern. We only consider braid patterns with a limited number of crossings, depending on the conformation of the companion knot.

Definition 5. We say a stick S in the cubic lattice has torsion $\frac{\pi}{2}$ if when we project to a perpendicular plane (i.e. “look down S ”), the projection of the adjacent sticks form an angle of $\frac{\pi}{2}$. We call S a *torsion stick*.

Note that every lattice stick conformation of a nontrivial knot cannot lie in a plane and therefore has at least two torsion sticks. Let J be a minimal lattice conformation of $[J]$. Let J_n be the conformation of the n component link formed by duplicating J $(n-1)$ times and translating each copy by the vector $(1, 1, -1)$ from the previous. We assume that we have scaled J up large enough so that no two copies intersect and so that all copies remain isotopic to one another in the resulting link complement.

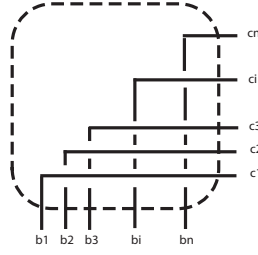
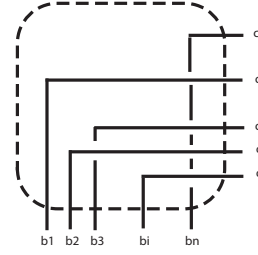
Definition 6. Define ω , a braid word, to be a *permutation word* if any two strings of w cross each other at most once and each crossing is in the same direction (all right over left or all left over right). The second condition is equivalent to the braid being either a positive braid or a negative braid.

Definition 7. We say a conformation of knot or link K *differs from J_n by a permutation word ω* if there exist n parallel sticks of J_n obtained from the translation process applied to a single stick of J which together yield a trivial braid, that when replaced by the permutation word ω yields K .

Proposition 4.1. *Satellite knots K that differ from J_n by a single permutation word ω have cubic lattice stick index $6nb[J]$.*

Proof. By [11], $b[K] \geq nb[J]$. Thus, $s_{CL}[K] \geq 6nb[J]$. Note that J_n has $6nb[J]$ sticks. We claim there is a transformation from J_n to K which preserves the number of sticks.

Note that the particular stick in J to which we apply the process of replacing the trivial braid given by the n corresponding parallel sticks in J_n by the braid given by ω is irrelevant, as all of the resulting knots will be equivalent. Since J is a knotted polygon in space and hence does not lie in a plane, there exists a torsion stick, say a . Label the two adjacent sticks b and c . Label the sticks in J_n corresponding to b and c as b_1, b_2, \dots, b_n and c_1, c_2, \dots, c_n respectively. Projecting down a , we see a picture as in Figure 4. The permutation word ω requires each b_i

FIGURE 4. Projection to the plane perpendicular to a .FIGURE 5. Permutation given $\omega(b_1) = c_i$ and $\omega(b_i) = c_1$.

to connect to some c_j . To do this, extend or shorten sticks b_i and c_j by $j - i$ units and connect them with a vertical stick. (See Figure 5.) This construction of K with $6nb[J]$ sticks gives us $s_{CL}[K] \leq 6nb[J]$. \square

Corollary 4.2. *Let S_n be the collection of knots which differ from J_n by at most one permutation word for each torsion stick of J . All knots in S_n have cubic lattice stick index $6nb[J]$.*

Proof. Since the algorithm in the previous proposition only changes the positions of one set of n sticks (the set corresponding to the original torsion stick), each torsion stick can allow K to differ from J_n by another permutation word. So all knots $K \in S_n$ can be constructed in $6nb[J]$ sticks. Therefore $s_{CL}[K] = 6nb[J]$ for $K \in S_n$. \square

We observe that taking satellites and composing knots are two operations that can be done independently, allowing us to determine cubic lattice stick index for a variety of different knots and knot classes. (See Figure 6).

5. CROSSING NUMBER BOUNDS

For a lattice stick conformation, let $|P_x|$, $|P_y|$, and $|P_z|$ denote the number of x , y and z sticks respectively. Following Huh and Oh in [8],

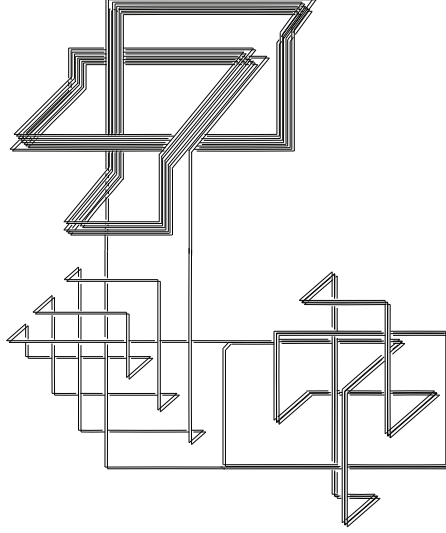


FIGURE 6. A 2-string satellite of (a 5-string satellite of 3_1 composed with the $(4, 5)$ -torus knot composed with a 2-string satellite of 9_{46}) with cubic lattice stick index 216.

we observe that we can always modify the lengths of sticks of a lattice knot conformation K , without changing the number of sticks, and translate the knot so that the vertices of K lie in the set $\{1, \dots, |P_x|\} \times \{1, \dots, |P_y|\} \times \{1, \dots, |P_z|\}$, and such that exactly two x -sticks have an endpoint at $x = i$ for any $i \in \{1, \dots, |P_x|\}$. We can do the same for the y -sticks and z -sticks concurrently. We call such a lattice knot conformation *properly leveled*. We can also extend this concept to links, the only distinction occurring when a trivial component lies entirely in a plane, say $x = k$. In this case we can arrange it so that *no* x -sticks have endpoints at the level $x = k$, but we will need more than $|P_x|$ levels. Suppose a lattice link conformation K has exactly C_x trivial components lying in planes perpendicular to the x -axis. Then we will call K *properly leveled* if the endpoints of x -sticks lie in a subset of the levels $x = 1, \dots, x = |P_x| + C_x$, where each level contains either two endpoints of x -sticks or a component of K , with similar conditions for y and z . As with lattice knots, we observe that every lattice link can be made properly leveled without altering the number of sticks in the conformation.

Remark 5.1. Let K be a properly leveled cubic lattice conformation of a link (in particular it could be a knot), and let P be the projection of K down the z -axis. Note that P is not necessarily a valid link diagram (i.e. a planar projection of a link conformation in \mathbb{R}^3 such that no

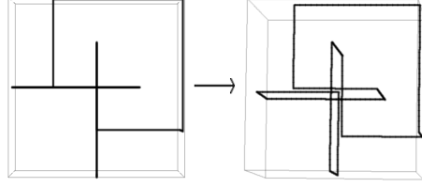


FIGURE 7. Perturbing a primary projection.

more than two points project to the same point, where double points are transverse intersections). Suppose that several x -sticks s_1, \dots, s_n all project to the same line $y = k$ in P . Unless there is a trivial component at $y = k$, there are exactly two y -sticks, t_a and t_b , which are connected to the level $y = k$. Let us slightly y translate s_1, \dots, s_n such that each lies on a different line, and such that the s_i 's are in the y order given by the z coordinates (see Figure 7). We can add very short line segments corresponding to the z -sticks and shorten or lengthen t_a and t_b , and continue this procedure for every set of line segments whose images under z -projection lie on the same line (using an analogous procedure to separate y -sticks) such that the result is a link diagram P' . Here P' is a z -projection of a slight perturbation of K (not necessarily remaining in the cubic lattice), made up of line segments corresponding to the sticks of K , where z -stick images are very short and are not involved in any crossings.

The advantage of P' is that as a link diagram we can use it to make statements about crossing number, linking number, and other invariants defined via link diagrams. We switch between P and P' as is convenient.

We now give bounds relating the cubic lattice stick index $s_{CL}[K]$ of a knot $[K]$ to its crossing index $c[K]$. We abbreviate $s_{CL}[K]$ and $c[K]$ by s_{CL} and c respectively.

Theorem 5.1. $s_{CL}[K] \geq 3\sqrt{c[K]} + 2$.

Proof. Let P and P' be projections of K down the z -axis, as in Remark 5.1. Since every crossing in P' arises from an intersection between an x -stick image and a y -stick image, we have immediately

$$c \leq (|P_x|)(|P_y|).$$

After a possible renaming we can assume $|P_x| \leq |P_z|$ and $|P_y| \leq |P_z|$. Then $|P_x| + |P_y| \leq \frac{2s_{CL}}{3}$, so $(|P_x|)(|P_y|)$ is certainly bounded from above

by the maximization of $(|P_x|)(|P_y|)$ constrained by

$$|P_x| + |P_y| = \frac{2s_{CL}}{3}.$$

This gives us

$$c \leq \frac{s_{CL}^2}{9}.$$

We can improve this bound slightly by noting that whenever an x -stick and a y -stick are directly attached or attached via a z -stick in between, there is no crossing in P' between these two sticks. This must occur at least twice in P' , so we have

$$c \leq \frac{s_{CL}^2}{9} - 2. \quad (5.1)$$

Inverting this inequality yields the result. \square

We now find an upper bound for s_{CL} as a function of c . Recall the following definitions as can be found in [2].

Definition 8. An *arc presentation* of a knot is an open-book presentation in which there are finitely many half-plane pages, such that each page meets the knot in a single simple arc. The *arc index* of a knot $[K]$, denoted $\alpha[K]$, or simply α , is the least number of pages in any arc presentation of $[K]$

Lemma 5.2. $s_{CL}[K] \leq 6\alpha[K] - 16$.

Proof. Let K be an minimal arc presentation with α pages having the z -axis as the binding axis. Ordering the pages based on their angle θ from the positive x -axis, isotope K such that the first three pages lie at $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$ respectively, and such that the remaining pages lie in $3\pi/2 < \theta < 5\pi/2$. The arc in the second page cannot connect to either of the arcs in the first and third page, as if it did, we could lower the arc index. Let P_a and P_b be the pages of the arcs that this arc does connect to.

Isotope the arcs of K lying in each of the first three pages without leaving the pages such that they each consist of three sticks, one parallel to the z -axis and two parallel to either the x -axis or y -axis (see Figure 8). Isotope page P_a to be the page given by $\theta = 2\pi$. The arc in this page can be realized by extending the connecting stick from the second page a distance α units and adding two other sticks in that half-plane, one z -stick, and one x -stick taking us back to the z -axis. The arc in plane P_b can be realized by extending another stick from the second plane across the z -axis into the half-plane given by $\theta = 2\pi$, and then adding

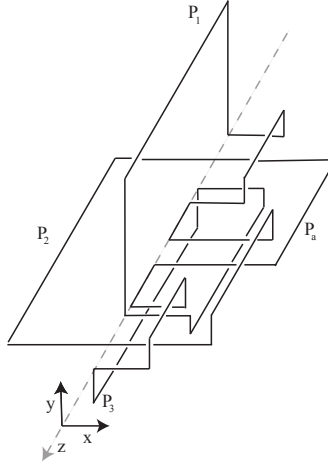


FIGURE 8. Constructing a cubic lattice conformation from an arc presentation.

four more sticks, one y -stick to leave the half-plane corresponding to $\theta = 2\pi$, one z -stick, one y -stick to return to the half-plane $\theta = 2\pi$ and one x -stick to return to the z -axis. The y -sticks will be to the negative y or positive y side of the $\theta = 2\pi$ plane depending on whether P_b comes before or after P_a in the sequence of half-planes. We can choose the length of the x -stick shorter than the x -stick in P_a to allow this arc to pass over or under the arc in P_a as needed.

Now all remaining arcs will start in the half-plane $\theta = 2\pi$, leaving that plane below or above it, depending on whether their original plane came before or after P_a in the sequence of pages. Orient the entire knot. For the two arcs that follow the arcs in the first and third page, we simply take an x -stick followed by a y -stick, a z -stick, a y -stick and an x -stick. For all other planes, we require a short z -stick together with the other sticks mentioned above. Finally, to close up at the end, we need another z -stick. We can pick the small z -sticks lying on the z -axis appropriately such that the result is a valid lattice conformation which is ambient isotopic to the original K .

Thus, if there are at least seven pages, we can construct a lattice stick version of K using $(3)(3) + 2 + 4 + 5 + 5 + 1 + 6(\alpha - 7) = 6\alpha - 16$ sticks. In the case that there are fewer than seven pages, the only knots generated are the trefoil knot and the figure-eight knot, both of which have lattice stick conformations that beat this bound.

□

From [1], we have $\alpha \leq c + 2$, which immediately yields the following corollary:

<i>Links</i>	s_{CL}
0_1^2	8
2_1^2	8
7_7^2	16
8_{15}^2	18
8_{16}^2	18
6_2^3	12
6_3^3	12

TABLE 1. Lattice stick index of certain links.

Corollary 5.3. $s_{CL}[K] \leq 6c[K] - 4$.

6. LINKS

In this section, we find the lattice stick number for certain links of more than one component.

Theorem 6.1. *The lattice stick number of certain links are as listed in Table 1.*

Proof. Note that the bound $s_{CL}[K] \geq 6b[K]$ holds only for nontrivial knots, which cannot lie entirely in a plane. As a consequence, this bound does *not* necessarily hold for arbitrary links, since one or more of the components may be contained in planes. However, if K_1, \dots, K_n are the components of a link L , we *do* have $s_{CL}[L] \geq s_{CL}[K_1] + \dots + s_{CL}[K_n]$. This lower bound together with the lattice conformations presented in Figure 9 complete the proof. \square

Proposition 6.2. *Let L be a two-component link conformation with linking number n and suppose that one of the components is entirely contained in a plane. Then the number of sticks in the conformation is at least $4n + 4$.*

Proof. Consider the projection P of L down the z -axis, and let P' be a slight perturbation of P as in Remark 5.1. As L has a linking number of n , so does P' . Let K_1 and K_2 denote the knot components of L and assume K_1 lies in a plane. Giving K_2 an orientation, we can therefore assume that K_2 crosses as an understrand from the inside to

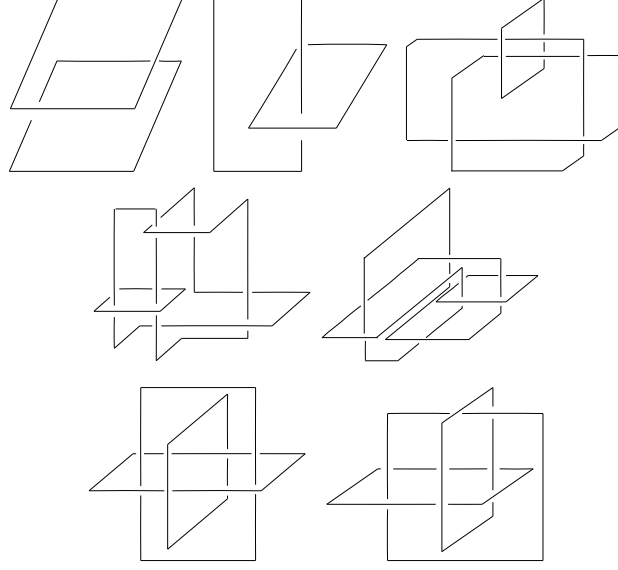


FIGURE 9. Minimal lattice conformations of links 0_1^2 , 2_1^2 , 7_7^2 , 8_{15}^2 , 8_{16}^2 , 6_2^3 , 6_3^3 .

the outside of K_1 at least n times and crosses as an overstrand from the outside to the inside of K_1 at least n times. Note that a single stick of K_2 makes at most a contributions of $1/2$ to the linking number of P' . Then K_2 requires at least $2n$ xy -sticks, each contributing $1/2$ to the linking number of P' , and a z -stick between consecutive xy -sticks in order to enable the crossings. \square

Corollary 6.3. *Let n be the linking number of a 2-braid link. Then the two-braid links, excluding 4_1^2 and 6_1^2 , can be realized with $4n + 4$ sticks.*

Proof. Note that the two exceptions have linking numbers 2 and 3, cases that are addressed in the next lemma. The Hopf link, with linking number 1, can obviously be realized with 8 sticks, and Figure 10 demonstrates how one can construct the other 2-braid links except 4_1^2 and 6_1^2 with $4n + 4$ sticks. \square

Lemma 6.4. *Let L be a two-component cubic lattice link conformation with linking number $n = 2$ or 3 and suppose that one of the components lies entirely in a plane. Then the number of sticks in the conformation is at least $4n + 5$.*

Proof. Assume for contradiction that L is composed of at most $4n + 4$ sticks, and as in the previous proof, let K_1 and K_2 denote the knot components of L , where K_1 lies entirely in the xy -plane and in the

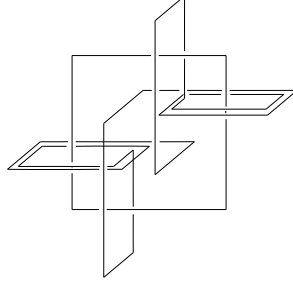


FIGURE 10. Two-braid link with linking number 6.

projection down the z -axis, that K_2 crosses as an understrand from the inside to the outside of K_1 at least n times and crosses as an overstrand from the outside to the inside of K_1 at least n times. Then K_2 requires at least $2n$ xy -sticks each contributing $1/2$ to the linking number of P' , and we need a z -stick between these consecutive xy -sticks in order to enable the crossings, amounting to an additional $2n$ sticks. We can then assume that K_1 is a rectangle made from four sticks, and the sticks of K_2 follow the repeating pattern: xy -stick passing over K_1 from outside to inside, descending z -stick, xy -stick passing under K_1 from inside to outside, ascending z -stick, xy -stick passing over K_1 from outside to inside, etc. A pair of xy -sticks connected by a z -stick projecting to outside of K_1 must both be either x -sticks or y -sticks.

Since this accounts for all $4n + 4$ sticks, the endpoints of all of the $2n$ xy -sticks that project to inside K_1 must all project to the same point, so as to avoid the necessity of additional xy -sticks to close up the knot. For convenience, assume that they project to the origin in the xy -plane. However then the z -sticks that connect these endpoints in pairs must all lie on the z -axis. But in order to obtain the requisite linking number, each interior endpoint of an understrand, which lies beneath the xy -plane, must connect by a z -stick to the interior endpoint of an overstrand, which lies above the xy -plane. However, at most one of the connecting z -sticks on the z -axis can pass through the origin, a contradiction. \square

Theorem 6.5. *The lattice stick index of 4_1^2 is 13.*

Proof. The linking number of 4_1^2 is $n = 2$. Suppose for contradiction that we have a cubic lattice conformation L of 4_1^2 with 12 sticks or less. Let K_1 and K_2 denote the knot components of L . Consider the projection P of L into the xy -plane, and let P' be a slight perturbation of P as in Remark 5.1.

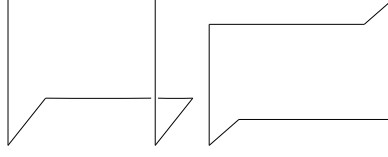
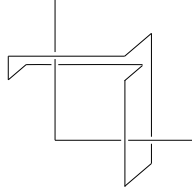


FIGURE 11. Six stick trivial components.


 FIGURE 12. Minimal lattice conformation of link 4_1^2 .

By Lemma 6.4, neither component may lie entirely in a plane. Therefore both K_1 and K_2 are constructed from 6 sticks. In particular, K_1 and K_2 are each constructed from 2 parallel sticks in each direction, and it is not hard to see that the two constructions shown in Figure 11 are the only possible 6-stick lattice conformations up to rotation and changing the lengths of sticks. We can assume that K_1 is positioned such that it z -projects to a rectangle. Then in the xy -plane it makes sense to talk about the inside and outside of K_1 , and as before we can assume K_2 crosses as an understrand from the inside to the outside of K_1 at least twice and crosses as an overstrand from the outside to the inside of K_1 at least twice. A single stick of K_2 can cross K_1 in P' at most twice, and such a stick may contribute 1 to the linking number. If we have such a stick, say x_{ij} , then there must be a parallel stick which either makes the opposite contribution to the linking number or does not contribute to the linking number (since there are two sticks in each direction and K_2 is a closed loop). Either case is clearly impossible, since then the remaining four sticks of K_2 lie in either the plane $x = i$ or $x = j$ and therefore cannot contribute to the linking number. We conclude that each of the four xy -sticks of K_2 must contribute $1/2$ to the linking number. All four inner endpoints must project to the same xy point in P , which is impossible without connecting oppositely oriented strands of K_2 or making K_2 into a two component link.

Figure 12 completes the proof.

□

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